

A-DISCRIMINANTS FOR COMPLEX EXPONENTS AND COUNTING REAL ISOTOPY TYPES

J. MAURICE ROJAS AND KORBEN RUSEK

INTRODUCTION

We take a first step toward extending the theory of \mathcal{A} -discriminants, and Kapranov's parametrization of \mathcal{A} -discriminant varieties [Kap91], to a broader family of functions including polynomials as a very special case. As an application, we prove a quadratic upper bound on the number of isotopy types of real zero sets of certain n -variate exponential sums, in a setting where the best previous bounds were strongly exponential in n . Our new topological bounds provide a significant improvement (albeit in a limited setting) over older bounds from fewnomial theory.

To be more precise, let $\mathcal{A} = [a_{i,j}] \in \mathbb{C}^{n \times t}$ have distinct columns, with j^{th} column a_j , $y := (y_1, \dots, y_n) \in \mathbb{C}^n$, $a_j \cdot y := \sum_{i=1}^n a_{i,j} y_i$, and consider exponential sums of the form

$$g(y) := \sum_{j=1}^t c_j e^{a_j \cdot y},$$

where $c_j \in \mathbb{C} \setminus \{0\}$ for all j . We call g an n -variate exponential t -sum, \mathcal{A} the spectrum of g , and let $Z_{\mathbb{R}}(g)$ denote the set of roots of g in \mathbb{R}^n . Also, given any two subsets $X, Y \subseteq \mathbb{R}^n$, an isotopy from X to Y (ambient in \mathbb{R}^n) is a continuous map $I : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying (1) $I(t, \cdot)$ is a homeomorphism for all $t \in [0, 1]$, (2) $I(0, x) = x$ for all $x \in \mathbb{R}^n$, and (3) $I(1, X) = Y$.¹ Although our generalized \mathcal{A} -discriminants do apply to arbitrary $\mathcal{A} \in \mathbb{C}^{n \times t}$, in this brief paper, our isotopy counts will focus on smooth $Z_{\mathbb{R}}(g)$, in the special case where $c_j \in \mathbb{R} \setminus \{0\}$ for all j , and $\mathcal{A} \in \mathbb{R}^{n \times (n+3)}$ and $s(g) := (\text{sign}(c_1), \dots, \text{sign}(c_t))$ (the sign vector of g) are fixed.

A consequence of our generalized \mathcal{A} -discriminants (defined in the next section, and parametrized explicitly in Theorem 1.7 below) is the following new count of isotopy types:

Theorem 1.1. *Following the notation above, assume $t = n + 3$, $\{a_1, \dots, a_{n+3}\}$ does not lie in any affine hyperplane, and that we fix \mathcal{A} and $s(g)$. Then the number of possible isotopy types for a smooth $Z_{\mathbb{R}}(g)$ is no greater than $2n^2 + 10n + 19$.*

Theorem 1.1 is proved in Section 3. Using the abbreviation $x^u := x_1^{u_1} \dots x_n^{u_n}$ when $(u_1, \dots, u_n) \in \mathbb{Z}^n$, we call $f(x) := \sum_{j=1}^t c_j x^{a_j}$ an n -variate t -nomial when $\mathcal{A} \in \mathbb{Z}^{n \times t}$. The change of variables $x = e^y := (e^{y_1}, \dots, e^{y_n})$ easily shows that, when $\mathcal{A} \in \mathbb{Z}^{n \times t}$, studying zero sets of t -nomials in the positive orthant \mathbb{R}_+^n is the same as studying zero sets of exponential t -sums in \mathbb{R}^n , up to the diffeomorphism between \mathbb{R}_+^n and \mathbb{R}^n defined by $x = e^y$. So dealing with exponential sums is indeed a generalization of the polynomial case.

Except for [BPRRR16] (which deals with the simpler cases $t \leq n + 2$), we are unaware of any earlier explicit upper bound for number of isotopy types of real zero sets of n -variate exponential t -sums. However, for the special case $\mathcal{A} \in \mathbb{Z}^{n \times (n+3)}$, and an additional restriction on the convex hull of $\{a_1, \dots, a_{n+3}\}$, [DRRS07, Thm. 1.3] gave an $O(n^6)$ upper bound. Also, a result of Basu and Vorobjov [BV07] implies a $2^{O(t^4)}(nt)^{O(t(t+n))}$ upper bound for arbitrary $\mathcal{A} \in \mathbb{Z}^{n \times t}$. One should also observe that if one allows $\{a_1, \dots, a_t\}$ to lie in an affine hyperplane,

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¹It is easily checked that an isotopy from X to Y implies an isotopy from Y to X as well. So isotopy is in fact an equivalence relation and it makes sense to speak of *isotopy type*.

then counting isotopy types becomes more subtle. In essence, this is because the number of isotopy types for smooth $Z_{\mathbb{R}}(g)$ with underlying spectrum $\mathcal{A} = [a_1, \dots, a_t]$ depends on dimension of the convex hull of $\{a_1, \dots, a_t\}$, as well as its cardinality.

Theorem 1.2. *If $\{a_1, \dots, a_t\}$ has cardinality $t \geq 2$ and convex hull of fixed dimension d , then the number of possible isotopy types for smooth $Z_{\mathbb{R}}(g)$ when g has fixed spectrum $[a_1, \dots, a_t] \in \mathbb{Z}^{n \times t}$ is t or $2^{\Omega(t)}$, according as $d=1$ or $d \geq 2$.*

Theorem 1.2 is proved in Section 4. Note that $Z_{\mathbb{R}}(g)$ is empty when $t=1$, and $1 \leq d \leq t-1$ when $t \geq 2$. Note also that the lower bounds for $d \geq 2$ assume $t \rightarrow \infty$ while d remains fixed. The cases where $t-d$ is fixed are thus *not* addressed by Theorem 1.2. In particular, the number of possible isotopy types for smooth $Z_{\mathbb{R}}(g)$, when \mathcal{A} and $s(g)$ are fixed, is known to be 2 when $t-d \leq 2$ (see, e.g., [BPRRR16]). Our Theorem 1.1 thus at last addresses the case $t-d=3$.

Let us now introduce our main theoretical tool.

1.1. Generalizing, and Parametrizing, \mathcal{A} -Discriminants for Complex Exponents.

Our isotopy count provides a motivation for generalized \mathcal{A} -discriminants, since discriminants parametrize degenerate behavior, and different isotopy types can be obtained by deforming a zero set through a degenerate state. This connection is classical and well-known (see, e.g., [GKZ94, Ch. 11, Sec. 5]): The connected components of the complement of the real part of a discriminant variety describe regions in coefficient space (called *discriminant chambers*) where the topology of the real zero set (in a suitable compactification of \mathbb{R}^n) of a polynomial is constant. Our central object of study is thus the following kind of discriminant variety associated to families of exponential sums:

Definition 1.3. *Let $\mathcal{A} \in \mathbb{C}^{n \times t}$ have j^{th} column a_j and assume $\hat{\mathcal{A}}$ has rank $n+1$. We then define the generalized \mathcal{A} -discriminant variety, $\Xi_{\mathcal{A}}$, to be the Euclidean closure of the set of all $[c_1 : \dots : c_t] \in \mathbb{P}_{\mathbb{C}}^{t-1}$ such that $\sum_{j=1}^t c_j e^{a_j \cdot z}$ has a degenerate root in \mathbb{C}^n . \diamond*

When $\mathcal{A} \in \mathbb{Z}^{n \times t}$ our $\Xi_{\mathcal{A}}$ agrees with the classical \mathcal{A} -discriminant variety $\nabla_{\mathcal{A}}$ [GKZ94]. More to the point, for most \mathcal{A} , $\nabla_{\mathcal{A}}$ is an algebraic hypersurface with a defining polynomial that is (too) often too unwieldy for computational purposes. So we will also need a more efficient, alternative description for $\Xi_{\mathcal{A}}$.

Example 1.4. *Let $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$. A routine maple calculation then shows us that*

$\nabla_{\mathcal{A}}$ *is the zero set of a polynomial $\Delta_{\mathcal{A}} \in \mathbb{Z}[c_1, \dots, c_5]$ satisfying $\Delta_{\mathcal{A}}(1, 1, 1, a, b) =$*

$$\begin{aligned} & 41987654504771523593992227a^8b^8 + 8568922617577790827960320a^8b^7 + 394594247668399678957787136a^7b^8 + \\ & 491069384583950065193975808a^8b^6 - 971141005960243113814917120a^7b^7 + 1644546811048059090366627840a^6b^8 + \\ & 557969223231079901560832a^9b^4 + 828434941582623838008508416a^8b^5 - 8896118143687124537286066176a^7b^6 + \\ & 4692084142913135619868721152a^6b^7 + 2845499698372999866809843712a^5b^8 + 557969223231079901560832a^4b^9 + \\ & 33392996500536631555522560a^9b^3 + 384254443547034707078152192a^8b^4 - 11483443502644561069909999616a^7b^5 + \\ & 14323107664774924348979937280a^6b^6 + 13591000063033685271054909440a^5b^7 + 2225676679631729339955937280a^4b^8 - \\ & 25511283567328457194995712a^3b^9 + 20941053496075364622925824a^9b^2 + 269737322421295126029533184a^8b^3 - \\ & 251458123743644571580497920a^7b^4 + 21319282121430982186963566592a^6b^5 + 18138163316374406659527671808a^5b^6 + \\ & 4594348961140867552012926976a^4b^7 + 38288951865122947982163968a^3b^8 - 51524645931445780035403776a^2b^9 + \\ & 363087263602825104457728a^{10} - 9792009640288689535844352a^9b + 178810349707236426746167296a^8b^2 + \\ & 1368264254117216589547831296a^7b^3 + 10397247952186084766590697472a^6b^4 + 6930726608820725492905672704a^5b^5 + \\ & 2535119422553880950892134400a^4b^6 + 134703665565747736152637440a^3b^7 - 92973237722754317832683520a^2b^8 - \\ & 2893351631835012551147520ab^9 + 363087263602825104457728b^{10} + 726174527205650208915456a^9 - \\ & 12696707749111290371506176a^8b - 74489621423517087836405760a^7b^2 + 363189381895713399018356736a^6b^3 - \\ & 482236618449489680142434304a^5b^4 + 19129025533750888626651136a^4b^5 + 50571247933680984080252928a^3b^6 - \\ & 31282237054780900405936128a^2b^7 - 5798049740657613386809344ab^8 + 726174527205650208915456b^9 + \\ & 363087263602825104457728a^8 - 2904698108822600835661824a^7b + 10166443380879102924816384a^6b^2 - \\ & 20332886761758205849632768a^5b^3 + 25416108452197757312040960a^4b^4 - 20332886761758205849632768a^3b^5 + \\ & 10166443380879102924816384a^2b^6 - 2904698108822600835661824ab^7 + 363087263602825104457728b^8. \diamond \end{aligned}$$

The *generalized* \mathcal{A} -discriminant need not be the zero set of *any* polynomial function when $\mathcal{A} \notin \mathbb{Z}^{n \times (n+3)}$: For instance, taking $\mathcal{A} = [0, 1, \sqrt{2}]$, it is not hard to check that the intersection of the $c_1 = c_3$ line with $\Xi_{\mathcal{A}}$ in $\mathbb{P}_{\mathbb{C}}^2$ is exactly the infinite set

$$\left\{ \left[1 : \frac{-\sqrt{2}}{\sqrt{2}-1}(\sqrt{2}-1)^{1/\sqrt{2}} e^{\sqrt{-2}\pi k} : 1 \right] \mid k \in \mathbb{Z} \right\}.$$

So this particular $\Xi_{\mathcal{A}}$ can't even be semi-algebraic.

Nevertheless, the generalized discriminant $\Xi_{\mathcal{A}}$ admits a concise and explicit parametrization that will be our main theoretical tool. Some notation we'll need is the following.

Definition 1.5. For any $\mathcal{A} \in \mathbb{C}^{n \times t}$ let $\hat{\mathcal{A}} \in \mathbb{C}^{(n+1) \times t}$ denote the matrix with first row $[1, \dots, 1]$ and bottom n rows forming \mathcal{A} , and assume there is a matrix $B \in \mathbb{C}^{t \times (t-n-1)}$ with columns forming a basis for the right nullspace of $\hat{\mathcal{A}}$. Let β_i denote the i^{th} row of B . Finally, let us call \mathcal{A} non-defective if we also have that (0) the columns of \mathcal{A} are distinct, (1) $\beta_i \neq \mathbf{0}$ for all i and (2) there are at least $t-n-1$ linearly independent $\ell \in \mathbb{C}^{t-n-1}$ such that $\sum_{[\beta_i]=[\ell]} \beta_i \neq \mathbf{0}$, where $[\ell]$ denotes the equivalence class of ℓ in $\mathbb{P}_{\mathbb{C}}^{t-n-2}$. \diamond

Note that the existence of a $B \in \mathbb{C}^{t \times (t-n-1)}$ with columns forming a basis for the right nullspace of $\hat{\mathcal{A}}$ is equivalent to $\hat{\mathcal{A}}$ having rank $n+1$. Also, \mathcal{A} being non-defective is a stronger condition: For instance, $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ has $\text{rank}(\hat{\mathcal{A}}) = 4$ but can be shown to be defective (see, e.g., [BHPR11, Ex. 2.8 & Cor. 3.7] or [DS02, DR06]). In general, since $[1, \dots, 1]$ is in the row-space of $\hat{\mathcal{A}}$, there are in fact at least $t-n$ *affinely* independent $\ell \in \mathbb{C}^{t-n-1} \setminus \{\mathbf{0}\}$ with $\sum_{[\beta_i]=[\ell]} \beta_i \neq \mathbf{0}$ when \mathcal{A} is non-defective.

Once we have a particular hyperplane arrangement it will be easy to write down our parametrization of $\Xi_{\mathcal{A}}$.

Definition 1.6. Let $(\cdot)^{\top}$ denote matrix transpose and, for any $u := (u_1, \dots, u_N)$ and $v := (v_1, \dots, v_N)$ in \mathbb{C}^N , let $u \odot v := (u_1 v_1, \dots, u_N v_N)$. Then, following the notation of Definition 1.5, assume \mathcal{A} is non-defective, set $\lambda := (\lambda_1, \dots, \lambda_{t-n-1})$, $[\lambda] := [\lambda_1 : \dots : \lambda_{t-n-1}]$, and let

$$R_{\mathcal{A}} := \left\{ \sum_{[\beta_i]=\ell} \beta_i \mid \ell \in \mathbb{P}_{\mathbb{R}}^{t-n-2} \right\} \setminus \{\mathbf{0}\}.$$

(So $R_{\mathcal{A}}$ consists of at most t nonzero vectors in \mathbb{R}^{t-n-1} , at least $t-n$ of which are affinely independent.) We then define the (projective) hyperplane arrangement

$$\mathcal{H}_{\mathcal{A}} := \{[\lambda] \mid \lambda \cdot \gamma = 0 \text{ for some } \gamma \in R_{\mathcal{A}}\} \subset \mathbb{P}_{\mathbb{C}}^{t-n-2},$$

and define $\xi_{\mathcal{A}} : (\mathbb{P}_{\mathbb{C}}^{t-n-2} \setminus \mathcal{H}_{\mathcal{A}}) \times \mathbb{C}^n \rightarrow \mathbb{P}_{\mathbb{C}}^{t-1}$ by $\xi_{\mathcal{A}}([\lambda], y) := [(\lambda B^{\top}) \odot e^{-yA}]$. \diamond

While $\xi_{\mathcal{A}}$ certainly depends on B , its image is in fact independent of B . Let $\text{Log}|\cdot| : \mathbb{C}^N \rightarrow \mathbb{R}^N$ denote the coordinate-wise log-absolute value map.

Theorem 1.7. If $\mathcal{A} \in \mathbb{C}^{n \times t}$ is non-defective then $\Xi_{\mathcal{A}}$ is the closure of the image of $\xi_{\mathcal{A}}$, and $\Xi_{\mathcal{A}}$ is the closure of a finite union of analytic hypersurfaces in $\mathbb{P}_{\mathbb{C}}^{t-n-2}$. In particular, if $t \geq n+3$, then $(\text{Log}|\Xi_{\mathcal{A}} B^{\top}|)B$ does not lie in any affine hyperplane.

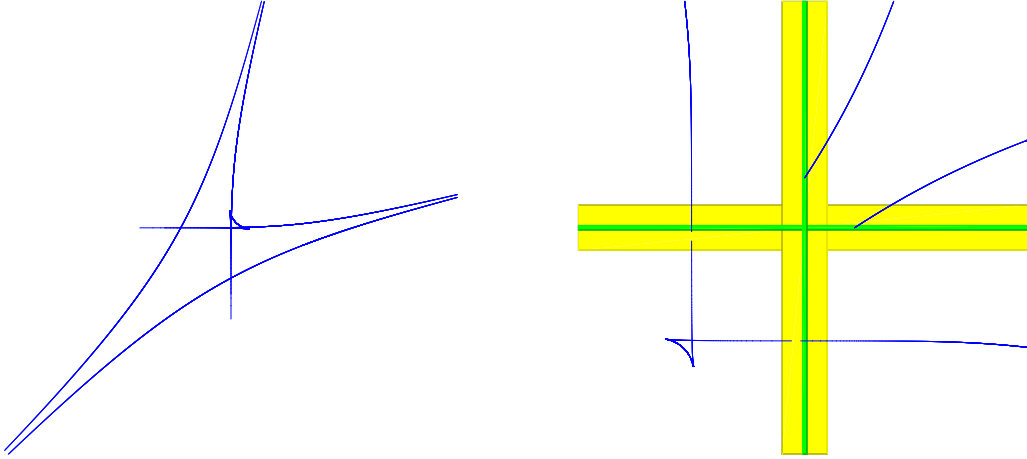
Theorem 1.7 is proved in Section 2. The special case $\mathcal{A} \in \mathbb{Z}^{n \times t}$ of Theorem 1.7 recovers the famous *Horn-Kapranov Uniformization*, derived by Kapranov in [Kap91]. The special case $\mathcal{A} \in \mathbb{R}^{n \times t}$ of Theorem 1.7 already appears to be new.

Since we'll ultimately be focussing on the real part of $\Xi_{\mathcal{A}}$ when \mathcal{A} is real, it will be especially useful to do one final change of coordinates on $\Xi_{\mathcal{A}}$. Toward this end, observe that $[1, \dots, 1]B = \mathbf{0}$. Clearly, for any $\ell \in (\mathbb{R} \setminus \{0\})^t$, we have that $(\text{Log}|\alpha\ell|)B$ is independent of α for any nonzero $\alpha \in \mathbb{C} \setminus \{0\}$, so we can then define $(\text{Log}|\ell|)B := (\text{Log}|\ell|)B$. Similarly, we can say that $\text{sign}([\ell]B) = (\sigma_1, \dots, \sigma_t)$ when there is a choice of $\varepsilon \in \{\pm 1\}$ with $\text{sign}(\varepsilon\ell B) = (\sigma_1, \dots, \sigma_t)$.

Definition 1.8. *Following the notation and assumptions of Definition 1.5, let $\varphi_{\mathcal{A}}(\lambda) := (\text{Log}|\lambda B^\top|)B$ for any $\lambda \in \mathbb{P}_{\mathbb{C}}^{t-n-2} \setminus \mathcal{H}_{\mathcal{A}}$. Also, assuming additionally that $\mathcal{A} \in \mathbb{R}^{n \times t}$ and $B \in \mathbb{R}^{t \times (t-n-1)}$ satisfy the hypotheses of Definitions 1.5 and 1.6, we then call $\mathcal{C}_{\mathcal{A}} := \varphi_{\mathcal{A}}(\mathbb{P}_{\mathbb{R}}^{t-1} \setminus \mathcal{H}_{\mathcal{A}})$ a reduced \mathcal{A} -discriminant contour of \mathcal{A} . Also, for any $\sigma \in \{\pm 1\}^t$, we call $\mathcal{C}_{\sigma, \mathcal{A}} := \{\varphi_{\mathcal{A}}(\lambda) \mid \text{sign}(\lambda B) = \sigma, \lambda \in \mathbb{P}_{\mathbb{C}}^{t-n-2} \setminus \mathcal{H}_{\mathcal{A}}\}$ a signed reduced \mathcal{A} -discriminant contour of \mathcal{A} . Finally, we call any connected component of the complement $\mathbb{R}^{t-n-1} \setminus \mathcal{C}_{\sigma, \mathcal{A}}$ a (signed reduced) \mathcal{A} -discriminant chamber. \diamond*

Note that given a non-defective real \mathcal{A} , we can easily find a real B satisfying all the assumptions above: For instance, we can simply use Gaussian Elimination. We will see later, in the proof of Theorem 1.10 below, that while $\mathcal{C}_{\sigma, \mathcal{A}}$ depends on B , all $\mathcal{C}_{\sigma, \mathcal{A}}$ with fixed (\mathcal{A}, σ) are equivalent up to an invertible linear map. So, while the $\mathcal{C}_{\sigma, \mathcal{A}}$ all clearly have dimension lower than the real part $\Xi_{\mathcal{A}} \cap \mathbb{P}_{\mathbb{R}}^{t-1}$ by $n-1$, the $\mathcal{C}_{\sigma, \mathcal{A}}$ still capture the topology of all smooth $Z_{\mathbb{R}}(g)$ with fixed \mathcal{A} and $s(g) = \sigma$.

Example 1.9.



Consider $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$ once again. The matrix $B = \begin{bmatrix} 4 & -4 & -1 & 1 & 0 \\ 4 & -1 & -4 & 0 & 1 \end{bmatrix}^\top$ satisfies the hypotheses of Definitions 1.5 and 1.6. The corresponding reduced contour, intersected with the square $[-12, 12]^2$ is drawn above on the left. On the right-hand illustration above, in the first through fourth quadrants, we see the reduced signed contours corresponding to the sign vectors $(+, +, +, +, +)$, $(+, +, +, -, +)$, $(+, +, +, -, -)$, $(+, +, +, +, -)$, intersected with the square $[-4, 4]$ and flipped suitably. (In particular, if one folds the right-hand illustration onto the first quadrant, then one recovers the left-hand illustration.) It then follows easily that for this \mathcal{A} , there are at most 3 reduced signed chambers for any fixed choice of sign vector in $\{\pm 1\}^5$. \diamond

Theorem 1.10. *Fix a non-defective $\mathcal{A} \in \mathbb{R}^{n \times t}$ and a $B \in \mathbb{R}^{t \times (t-n-1)}$ satisfying the hypotheses of Definition 1.8, and assume both $c = (c_1, \dots, c_t), c' = (c'_1, \dots, c'_t) \in \mathbb{R}^t$ have sign vector σ , and $(\text{Log}|c|)B$ and $(\text{Log}|c'|)B$ lie in the same signed reduced \mathcal{A} -discriminant chamber. Then,*

assuming every facet of the convex hull $\text{Conv}\{a_1, \dots, a_{n+3}\}$ contains exactly $n+1$ points of \mathcal{A} , we have that $Z_{\mathbb{R}}\left(\sum_{j=1}^t c_j e^{a_j \cdot y}\right)$ and $Z_{\mathbb{R}}\left(\sum_{j=1}^t c'_j e^{a_j \cdot y}\right)$ are isotopic. ■

Theorem 1.10 follows easily from classical Morse Theory upon observing that there can be no singularities at infinity under the facet assumption for \mathcal{A} . (See, e.g., [GKZ94, Ch. 11, Sec. 5] for the special case $\mathcal{A} \in \mathbb{Z}^{n \times t}$.) We may remove the facet assumption at the expense of refining our chambers and increasing their number slightly. This is made precise in the proof of Theorem 1.1 below.

One may wonder why we introduced $\text{Log}|\cdot|$ in our definition of reduced \mathcal{A} -discriminant contour, since $\text{Log}|\cdot|$ does not affect connectivity when restricting to a fixed orthant. An important technical reason is that $\text{Log}|\cdot|$ makes the curvature of the image of φ much easier to study.

Theorem 1.11. *At any point $[\ell] \in \mathbb{P}_{\mathbb{R}}^{t-n-2} \setminus \mathcal{H}_{\mathcal{A}}$ where φ is differentiable, we have that $v \in \mathbb{R}^{t-n-1} \setminus \{\mathbf{O}\}$ is a normal vector to $\varphi_{\mathcal{A}}([\ell]) \iff [v] = [\ell]$. ■*

The special case $\mathcal{A} \in \mathbb{Z}^{n \times t}$ of Theorem 1.11 was already observed by Kapranov in [Kap91, Thm. 2.1 (b)]. The proof of the more general Theorem 1.11 in fact follows almost identically, since it ultimately reduces to elementary identities involving linear combinations of logarithms in linear forms where one merely needs \mathcal{A} to be real.

We are now ready to prove our main results.

2. THE PROOF OF OUR PARAMETRIZATION: THEOREM 1.7

Let $c := (c_1, \dots, c_t) \in (\mathbb{C} \setminus \{0\})^t$. Rewriting the equations defining g having a singularity at $y \in \mathbb{C}^n$, we see that g has a singularity at y if and only if

$$\widehat{\mathcal{A}} \begin{bmatrix} c_1 e^{a_1 \cdot y} \\ \vdots \\ c_t e^{a_t \cdot y} \end{bmatrix} = \mathbf{O}.$$

The last equality is equivalent to $c \odot e^{yA} = \lambda B^{\top}$ for some $\lambda \in (\mathbb{C}^*)^{t-n-1}$, thanks to the definition of the right nullspace of $\widehat{\mathcal{A}}$. (Note that we also obtain that λ must be such that no coordinate of λB^{\top} vanishes.) Dividing the i^{th} coordinate of each side by $e^{a_j \cdot y}$, we thus obtain that the image $\xi_{\mathcal{A}}((\mathbb{P}_{\mathbb{C}}^{t-n-2} \setminus \mathcal{H}_{\mathcal{A}}) \times \mathbb{C}^n)$ is exactly $\Xi_{\mathcal{A}} \cap [(\mathbb{C} \setminus \{0\})^t]$. Notice also that $\xi_{\mathcal{A}}$ is an analytic map defined by a multivariate exponential sum with coefficients that are linear polynomials. So by o -minimality [vdD98], the singular locus of its image admits a finite stratification, and we thus obtain that $\Xi_{\mathcal{A}} \cap [(\mathbb{C} \setminus \{0\})^t]$ is a union of finitely many analytic hypersurfaces. Taking closures, we are done, save for the final assertion about $\Xi_{\mathcal{A}}$ not lying in some affine hyperplane.

To prove this final assertion, observe that since \mathcal{A} is non-defective, $\mathcal{R}_{\mathcal{A}} \subset \mathbb{R}^{n-t-1}$ contains affinely independent (nonzero) vectors $\bar{\beta}_1, \dots, \bar{\beta}_r$ with $r \geq t-n$. In particular, by fixing a point p_i on the hyperplane $\{\bar{\beta}_i \cdot \lambda = 0\}$ (avoiding every other hyperplane $\{\bar{\beta}_j \cdot \lambda = 0\}$ for $j \neq i$) for each $i \in \{1, \dots, t-n\}$, the singularities of our parametrization $\xi_{\mathcal{A}}$ imply that $\varphi_{\mathcal{A}}(\Xi_{\mathcal{A}})$ contains points approaching rays of the form $\alpha_i + \mathbb{R}_+ \bar{\beta}_i$ for each $i \in \{1, \dots, t-n\}$. In other words, it is impossible for $\varphi_{\mathcal{A}}(\Xi_{\mathcal{A}})$ to lie in an affine hyperplane. ■

3. COUNTING ISOTOPY TYPES: PROVING THEOREM 1.1

We will first need the following convenient characterization of the singularities of a reduced contour when $t = n+3$.

Lemma 3.1. *If $\mathcal{A} \in \mathbb{R}^{n \times (n+3)}$ is non-defective then any reduced \mathcal{A} -discriminant contour has no more than n cusps.*

Sketch of Proof: Considering $\xi_{\mathcal{A}}([\lambda_1 : \lambda_2]) = (\xi_1([\lambda_1 : \lambda_2]), \xi_2([\lambda_1 : \lambda_2]))$ and dehomogenizing by setting $(\lambda_1, \lambda_2) = (1, \lambda)$, we can detect cusps by setting $\frac{\partial \xi_1}{\partial \lambda} = \frac{\partial \xi_2}{\partial \lambda} = 0$. One then obtains a pair of equations of the form $\sum_{i=1}^{n+3} \frac{\gamma_i}{\beta_{i,1} + \beta_{i,2}\lambda} = 0$. Clearing denominators, we then obtain a polynomial of degree $\leq n+2$. One can then check that the two leading coefficients are exactly 0, and thus we in fact obtain a polynomial of degree $\leq n$. So we clearly obtain that no more than n points of the form $[\lambda_1 : \lambda_2] \in \mathbb{P}_{\mathbb{R}}^1$ can yield a cusp. ■

In 1826, Steiner studied line arrangements in \mathbb{R}^2 and proved that m lines determine no more than $\frac{m(m-1)}{2} + m + 1$ connected components for the complement of their union [Ste26]. Thanks to our development so far, our reduced contour consists of a union of (possibly singular) arcs C_1, \dots, C_{n+3} . So we will need an analogue of Steiner's count for this non-linear arrangement. Fortunately, Theorem 1.11 and Lemma 3.1 imply that we can easily derive such an analogue. In particular, Theorem 1.11 implies that no two distinct smooth sub-arcs from the C_i can intersect more than once. The presence of cusps then implies a few more intersections. Employing Theorem 3.6 (and the proof of Theorem 3.7) from [Rus13], we then obtain the following:

Theorem 3.2. *Suppose $C_1, \dots, C_{n+3} \subset \mathbb{R}^2$ is a collection of piece-wise smooth arcs with a total of ℓ cusps. Suppose also that no two distinct smooth sub-arcs from the C_i can intersect more than once, and that consecutive sub-arcs from the same C_i do not intersect. Then $C_1 \cup \dots \cup C_{n+3}$ has at most $\frac{(n+\ell+3)(n+\ell+2)}{2} - (n+\ell+3)$ intersections, and the complement $\mathbb{R}^2 \setminus (C_1 \cup \dots \cup C_{n+3})$ has at most $\frac{(n+\ell+3)(n+\ell+2)}{2} - \ell + 1$ connected components. ■*

Combining Lemma 3.1 and Theorem 3.2 we then obtain the following:

Corollary 3.3. *For any non-defective $\mathcal{A} \in \mathbb{R}^{n \times (n+3)}$, the complement of the reduced \mathcal{A} -discriminant contour consists of no more than $2n^2 + 4n + 4$ connected components. ■*

The final fact we'll need before proving Theorem 1.1 is a refinement of Theorem 1.1 involving $\mathcal{A} \in \mathbb{R}^{n \times (n+2)}$:

Lemma 3.4. *Assume $\{a_1, \dots, a_{n+2}\} \subset \mathbb{R}^n$ does not lie in any affine hyperplane and assume $\mathcal{A} = [a_1, \dots, a_{n+2}]$ and $B \in \mathbb{R}^{(n+2) \times 2}$ satisfy the hypotheses of Definition 1.6. Then the image of the intersection of any orthant of $\mathbb{P}_{\mathbb{R}}^{n+1}$ with $\Xi_{\mathcal{A}}$ under $\varphi_{\mathcal{A}}$ is either empty or a fixed affine hyperplane depending only on \mathcal{A} . ■*

The proof of Lemma 3.4 is simply a stream-lining of the proof of Theorem 1.7, based on the fact that the reduced \mathcal{A} -discriminant contour is just a point when $\mathcal{A} \in \mathbb{R}^{n \times (n+2)}$ has $\text{rank}(\widehat{\mathcal{A}}) = n + 1$.

Sketch of Proof of Theorem 1.1: The special case where every facet of $\text{Conv}\{a_1, \dots, a_{n+3}\}$ contains exactly $n+1$ points of \mathcal{A} follows immediately from Corollary 3.3 and Theorem 1.10. So we need to remove the facet assumption.

Toward this end, assume $\text{Conv}\{a_1, \dots, a_{n+3}\}$ has a facet Q containing $n+2$ distinct points of $\{a_1, \dots, a_{n+3}\}$. Without loss of generality, we may assume a_{n+3} is not in Q and, by an invertible linear change of variables, we may also assume that $a_{n+3} = e_n$ (the n^{th} standard basis vector) and a_1, \dots, a_{n+2} lie in the $y_n = 0$ hyperplane. It then follows easily that $Z_{\mathbb{R}}(f)$

can have no singularities in \mathbb{R}^n and that any change in the topology of $Z_{\mathbb{R}}(g)$ must occur at infinity (in the limit as $y_n \rightarrow -\infty$). In particular, we obtain that $\Xi_{\mathcal{A}} = \Xi_{\{a_1, \dots, a_{n+2}\}}$. So we conclude by downward induction on n , assuming we can handle the remaining case correctly.

The remaining case is where $\text{Conv}\{a_1, \dots, a_{n+3}\}$ has at least one facet containing exactly $n+1$ distinct points of $\{a_1, \dots, a_{n+3}\}$. In this case, any such facet can also enable singularities to occur at infinity (in addition to possible singularities in \mathbb{R}^n), but these singularities are controlled by $\Xi_{\mathcal{A}'}$ where, up to a linear change of variables, \mathcal{A}' is in the setting of Lemma 3.4. So if we refine our signed reduced chambers by cutting our signed reduced contour with affine lines (one for each facet possessing exactly $n+1$ distinct points of $\{a_1, \dots, a_{n+3}\}$), then it is enough to show that the total number of chambers is still no greater than our stated bound. Considering how these facets can intersect, it easily follows that there can be at most 3 such facets. By Theorem 1.11 and Lemma 3.1, our signed reduced contour is a union of at most $2n+3$ convex arcs. So cutting by the first line introduces at most $2n+4$ new connected components. The next two lines then clearly introduce at most $2n+5$ and $2n+6$ new connected components. So in the end, we have at most

$$2n^2 + 4n + 8 + (2n+4) + (2n+5) + (2n+6) = 2n^2 + 10n + 19$$

connected components where $Z_{\mathbb{R}}(g)$ has constant topology for a fixed signed vector. So we are done. ■

4. PROOF OF ASYMPTOTIC ISOTOPY LOWER BOUNDS: THEOREM 1.2

By an invertible linear change of variables, we may assume that $d=n$. Substituting $x_i = e^{y_i}$ as usual, we then see that the case $d=1$ is nothing more than counting the possible numbers of positive roots of a univariate t -nomial (since we've assumed $\mathcal{A} \in \mathbb{Z}^{n \times t}$). Via a classic refinement of Descartes' Rule (see, e.g., [Gra99]), such a t -nomial can attain k roots for any $k \in \{0, 1, \dots, t-1\}$ and we thus obtain the exact count of t for the number of isotopy types of g . (In fact, this isotopy count persists even if the exponents are real.)

The case $d \geq 2$ follows via Viro's Patchworking. In particular, in [OK03], it is proved that the number of isotopy types for a real projective hypersurface defined by a homogeneous $(d+1)$ -variate polynomial of degree r is $2^{\Omega(r^d)}$. This construction in fact yields asymptotically the same number of isotopy types for a hypersurface in \mathbb{R}_+^d defined by an d -variate polynomial of degree r , since we can use translation of the variables to move any ovals into the positive orthant. The number of monomial terms of such a polynomial is $\frac{(r+d) \cdots (r+1)}{d!} = \Theta(r^d)$ for fixed d and increasing r . So this family of polynomials evinces $2^{\Omega(t)}$ distinct isotopy types for positive zero sets. Substituting exponentials for the variables, we are done. ■

Remark 4.1. *We have so far not addressed the conceptually simpler question of bounding the number of connected components of $Z_{\mathbb{R}}(g)$. For the sake of brevity, let us just point out that the number of isotopy types can be far larger than the maximal number of connected components. For instance, in our preceding proof, we see already that degree r curves have $2^{\Omega(r^2)}$ isotopy types for their positive zero sets, but the maximal number of connected components of such a zero set is easily seen to be $O(r^2)$ by translating ovals and an application of Harnack's classical estimate. ([Vir08] contains an elegant discussion of Harnack's estimate.) ◇*

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E-mail address: rojas@math.tamu.edu

TAMU 3368, COLLEGE STATION, TX 77843-3368

E-mail address: korben.rusek@gmail.com

1311 H ST SE, AUBURN, WA 98002, USA